PUTNAM PRACTICE SET 24: SOLUTIONS

PROF. DRAGOS GHIOCA

Problem 1. Find a polynomial $P(x,y) \in \mathbb{R}[x,y]$ with the property that for each real number r, we have

$$P\left([r],[2r]\right) = 0,$$

where $[x]$ is always the integer part of the real number x (i.e., the largest integer less than or equal to x).

Solution. We let

$$P(x,y) = (y-2x)\cdot(y-2x-1)$$

and note that for each real number r, we have that

$$\text{either } [2r] = 2\cdot[r], \text{ or } [2r] = 2[r]+1,$$

which means that $P([r],[2r]) = 0$ for each $r \in \mathbb{R}$.

Problem 2. Show that the curve in the cartesian plane given by the equation:

$$x^3 + 3xy + y^3 = 1$$

contains exactly one set of three points A, B and C which are the vertices of an equilateral triangle.

Solution. The whole key to this problem is the following factorization:

$$x^3 + y^3 + 3xy - 1 = (x+y-1)(x^2+y^2+1-xy+x+y)$$

which comes from the identity:

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx).$$

Now, using the fact that

$$x^2+y^2+1-xy+x+y = \frac{1}{2}\cdot(x-y)^2 + \frac{1}{2}\cdot(x+1)^2 + \frac{1}{2}\cdot(y+1)^2,$$

we get that besides the line $x+y=1$, the given plane curve contains *only* the point $(-1,-1)$. So, indeed, there is only one triple of points on the given curve which are the vertices of an equilateral triangle; one of those three points must be $(-1,-1)$, while the other two points lie on the line $x+y=1$ being exactly $\frac{1}{\sqrt{3}}\cdot h$ units apart from the point $\left(\frac{1}{2},\frac{1}{2}\right)$, which is the foot of the perpendicular line from $(-1,-1)$ to the line $x+y=1$, where h is the length of the height from $(-1,-1)$ to this line, i.e.,

$$h = \sqrt{2}\cdot\frac{3}{2}.$$

So, the other two vertices of the equilateral triangle are

$$\left(\frac{1+\sqrt{3}}{2},\frac{1-\sqrt{3}}{2}\right) \text{ and } \left(\frac{1-\sqrt{3}}{2},\frac{1+\sqrt{3}}{2}\right).$$

Problem 3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers satisfying the two properties:

$$a_i = i \text{ for } i = 1, \dots, 2020 \text{ and } a_n = a_{n-1} + a_{n-2020} \text{ for } n \geq 2021.$$

Show that for each positive integer M, there exists some integer $k > M + 2020$ such that each one of the integers $a_k, \dots, a_{k+2018}$ are divisible by M.

Solution. We extend the definition of the sequence $\{a_n\}$ for *all* $n \in \mathbb{Z}$ simply by enforcing the condition

$$a_n = a_{n-1} + a_{n-2020}$$

for all $n \in \mathbb{Z}$. Note that we can solve for a_0 from

$$a_{2020} = a_{2019} + a_0$$

and get $a_0 = 1$. Similarly, we solve for a_{-1} from

$$a_{2019} = a_{2018} + a_{-1}$$

and get $a_{-1} = 1$. Furthermore, $a_{-k} = 1$ for each $k \in \{0, 1, \dots, 2018\}$. Then we have $a_{-2019} = 0$ because

$$a_1 = a_0 + a_{-2019}$$

and $a_1 = a_0 = 1$. Continuing to solve backwards, we get

$$a_{-k} = 0 \text{ for } k = 2019, 2020, \dots, 4037.$$

For example, note that

$$a_{-2017} = a_{-2018} + a_{-4037}$$

and $a_{-2017} = a_{-2018} = 1$.

Therefore, there exist 2019 consecutive integers in our recurrence sequence, all of them equal to 0.

On the other hand, for any given positive integer M, any recurrence sequence is eventually periodic modulo M. Furthermore, since for our sequence we can solve also backwards (as shown above), the sequence is *actually* periodic modulo M. (**The same trick can be applied to the Fibonacci sequence, for example, to show that for any integer M there exist infinitely many terms in the Fibonacci sequence all of them divisible by M.**)

So, since at one point we had 2019 consecutive integers in our sequence all divisible by M (simply because those integers are all equal to 0), then we can find such consecutive integers divisible by M in our sequence with indices arbitrarily large.

Just to give more details to our reasoning: first of all, since there exist finitely many residue classes modulo M (for any given positive integer M), there must exist two distinct tuples of 2020 *consecutive elements in our sequence which give us the same residue classes modulo M. So, there exist two distinct* 2020 *consecutive tuples of elements in our sequence*

$$(a_k, a_{k+1}, \dots, a_{k+2019}) \text{ and } (a_\ell, a_{\ell+1}, \dots, a_{\ell+2019})$$

such that $a_{k+i} \equiv a_{\ell+i} \pmod{M}$ for each $i = 0, 1, \dots, 2019$, then our linear recurrence formula yields that

$$a_{k+2020} \equiv a_{k+2019} + a_k \equiv a_{\ell+2019} + a_\ell \equiv a_{\ell+2020} \pmod{M}$$

and more generally, inductively, we get that for each nonnengative integer i, we have that

$$a_{k+i} \equiv a_{\ell+i} \pmod{M}.$$

But also, going backwards, we have

$$a_{k-1} \equiv a_{k+2019} - a_{k+2018} \equiv a_{\ell+2019} - a_{\ell+2018} \equiv a_{\ell-1} \pmod M$$

and then also, for all $i \in \mathbb{N}$, we have

$$a_{k-i} \equiv a_{\ell-i} \pmod M,$$

thus showing that our linear recurrence sequence is periodic modulo M. Since at one moment (for the indices $k = -2019, -2020, \dots, -4037$) we have 2019 consecutive integers in our sequence all divisible by M (since in that case, they're all equal to 0), then the same phenomenon repeats infinitely often, i.e., there exist arbitrarily large positive integers k such that $a_k, a_{k+1}, \dots, a_{k+2018}$ are all divisible by M, as desired.

Problem 4. Let n be a positive integer and let $\theta \in \mathbb{R}$ such that θ/π is an irrational number. For each $k = 1, \dots, n$, we let

$$a_k = \tan\left(\theta + \frac{k\pi}{n}\right).$$

Compute $\frac{a_1+a_2+\dots+a_n}{a_1 \cdot a_2 \cdot \dots \cdot a_n}$.

Solution. We let

$$\omega := e^{2\theta n \cdot i} = \cos(2n\theta) + i\sin(2n\theta).$$

For the polynomial

$$P(x) = (1+ix)^n - \omega \cdot (1-ix)^n,$$

we compute for each $k = 1, \dots, n$ that

$$P(a_k) = \left(\frac{\cos\left(\theta + \frac{k\pi}{n}\right) + i\sin\left(\theta + \frac{k\pi}{n}\right)}{\cos\left(\theta + \frac{k\pi}{n}\right)}\right)^n - \omega \cdot \left(\frac{\cos\left(\theta + \frac{k\pi}{n}\right) - i\sin\left(\theta + \frac{k\pi}{n}\right)}{\cos\left(\theta + \frac{k\pi}{n}\right)}\right)^n$$

and so, letting

$$\varepsilon_k := e^{(n\theta + k\pi)\cdot i},$$

we see that

$$P(a_k) = \frac{\varepsilon_k - \omega \cdot \bar{\varepsilon_k}}{\cos^n\left(\theta + \frac{k\pi}{n}\right)} = 0$$

because

$$\frac{\varepsilon_k}{\bar{\varepsilon_k}} = e^{2n\theta \cdot i} = \omega.$$

In conclusion, the polynomial $P(z)$ vanishes at each point a_k for $k = 1, \dots, n$ and since it also has degree n and leading coefficient equal to

$$c_n := i^n - \omega \cdot (-i)^n,$$

we conclude that

$$P(z) = c_n \cdot \prod_{k=1}^n (z - a_k).$$

So,

$$\frac{\sum_{k=1}^n a_k}{\prod_{k=1}^n a_k} = \frac{-c_{n-1}}{(-1)^n c_0},$$

where we write

$$P(z) = c_n z^n + c_{n-1} z^{n-1} + \dots + c_1 z + c_0.$$

Clearly,

$$c_0 = 1 - \omega \text{ and } c_{n-1} = ni^{n-1} - \omega \cdot n(-1)^{n-1} i^{n-1} = ni^{n-1} \cdot \left(1 + \omega(-1)^n\right),$$

which means that

$$\frac{\sum_{k=1}^{n} a_k}{\prod_{k=1}^{n} a_k} = \frac{1 + \omega(-1)^n}{1 - \omega} \cdot n(-i)^{n-1}.$$

As a *fun* fact, if n is *odd*, then the above quotient is always an integer because then $1 + \omega(-1)^n = 1 - \omega$.